TWMS J. Pure Appl. Math., V.11, N.1, 2020, pp.30-42

TRANSPORTATION OF A BVP INCLUDING GENERALIZED CAUCHY-RIEMANN EQUATION TO FREDHOLM INTEGRAL EQUATION

J. EBADPOUR GOLANBAR[1], N. ALIEV[2], M. JAHANSHAHI[1]

Abstract. In this paper, a boundary value problem including generalized nonhomogenious Cauchy-Riemann equation with general nonlocal boundary conditions in a bounded planer region with Lyapunov line boundary is investigated. First, some states called compatibility conditions are obtained by making use of the generalized solution of Cauchy-Riemann equation. Then, the second kind of Fredholm integral equation for boundary values of the unknown of the main problem is resulted by applying and giving boundary condition. Finally, the singularities in integral kernels are removed or reduced to weak singularities by making use of the proposed method in this paper.

Keywords: boundary value problem, generalized Cauchy-Riemann equation, compatibility conditions, Fredholm integral equation, regularization, weak singularities

AMS Subject Classification: 45B05, 35J56, 35F15, 65N80.

1. Introduction

Elliptic equations have a fundamental role in the theory of boundary value problems. The most important of these equations are Laplace equation, biharmonic, Helmholtz equation, Cauchy-Riemann equation and others. [8, 13, 21]. According to applications of elliptic equations in physics and engineering, many mathematicians and scientists have investigated and solved these problems via analytical and numerical classical methods, see [14, 17, 21]. The second and third authors studied Poisson equation, Cauchy-Riemann equation and Navier-Stokes system of equations with local and non-local boundary conditions, respectively in [2, 3, 9]. The proposed method is based on reducing the given boundary value problem to the second kind of Fredholm boundary integral equations and removing singularities in the kernel of integral expressions [4, 6, 10, 15, 19]. In this paper, we consider the generalized nonhomogenous elliptic Cauchy-Riemann equation:

$$\ell u \equiv \frac{\partial u(X)}{\partial x_2} + i\frac{\partial u(X)}{\partial x_1} + a(X)u(X) = f(X)$$
$$X = (x_1, x_2) \in D \subset R^2,\ (\sqrt{-1} = i), \tag{1}$$

with general non-local boundary condition

$$\ell_1 u \equiv \alpha_1(x_1)u\left(x_1, \gamma_1(x_1)\right) + \alpha_2(x_1)u\left(x_1, \gamma_2(x_1)\right) = \alpha(x_1), \qquad x_1 \in [a_1, b_1], \tag{2}$$

[1]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
[2]Institute of Applied Mathematics, Baku State University, Baku Azerbaijan
e-mail: j.ebadpour@azaruniv.ac.ir, jahanshahi@azaruniv.ac.ir
Manuscript received August 2017.

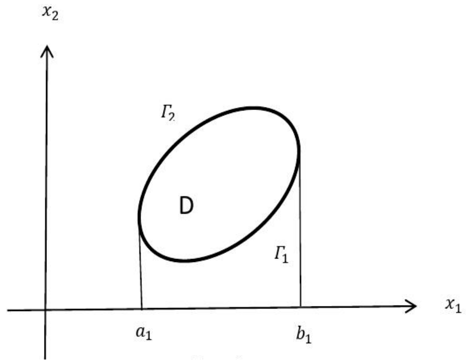

Figure 1.

where $\partial D = \Gamma = \Gamma_1 \cup \Gamma_2$ which $\Gamma_k : x_2 = \gamma_k(x_1)$, $k = 1, 2$ and $\gamma_1(a_1) = \gamma_2(a_1)$, $\gamma_1(b_1) = \gamma_2(b_1)$ as show in Figure 1. $[a_1, b_1] = proj|\Gamma_1 = proj|\Gamma_2$. And the functions $a(X)$ and $f(X)$ are continuous functions on $\overline{D}$, $\alpha(x_1), \alpha_i(x_1)$, $i = 1, 2$ are known and continuous functions on the interval $[a_1, b_1]$. The rest of this article will be presented as follows. In the second section, by using the generalized (fundamental) solution of Cauchy-Riemann equation and by applying Ostrogradsky formula and also by using Delta-Dirac function and its properties, the compatibility conditions are obtained. Also, in this section by making use of the resulted compatibility conditions and given boundary conditions, we obtain integral expressions for the boundary values of unknown $u(x_1, x_2)$ which they have singularities. The third section is devoted to the regularization. In this section, singularities in kernels of integral expressions are removed or reduced to weak singularities. In the final section, main results will be given, especially, the given BVP is transformed to the second kind Fredholm boundary integral equations.

Remark 1.1. *According to Fredholm theory, when the given BVP under hypothesis on data of problem reduces to a second kind Fredholm integral equation, we can state the existence and uniqueness of solution of BVP based on Fredholm alternative theorems due to parameter* λ [13].

Remark 1.2. *The boundary conditions (2) is in case of general linear nonlocal boundary condition. By suitable select of coefficients* $\alpha_i(x_1)$, $i = 1, 2$ *we can get Dirichlet classical boundary condition.*

2. Compatibility conditions

For getting compatibility conditions (or necessary conditions), first consider the fundamental solution of the Cauchy-Riemann equation
$\dfrac{\partial u(X)}{\partial x_2} + i\dfrac{\partial u(X)}{\partial x_1} = 0.$ [20]:

$$U(X - \xi) = \frac{1}{2\pi} \cdot \frac{1}{(x_2 - \xi_2) + i(x_1 - \xi_1)}. \quad (3)$$

If this solution is multiplied to both side of equation (1) and integrated over the domain D, then we have:

$$\begin{aligned} (\ell u, U) \quad &\equiv \quad \int_D \left[\frac{\partial u(X)}{\partial x_2} + i\frac{\partial u(X)}{\partial x_1} + a(X)u(X) \right] U(X - \xi)]dX \\ &= \quad \int_D f(X)U(X - \xi)dX. \end{aligned}$$

After that by using Ostrogradsky formula and Green second theorem we have:

$$\begin{aligned}
&\int_{\Gamma} u(X)U(X-\xi)\big[\cos(\nu,x_2)+i\cos(\nu,x_1)\big]dX \\
&-\int_{D} u(X)\left[\frac{\partial U(X-\xi)}{\partial x_2}+i\frac{\partial U(X-\xi)}{\partial x_1}\right]dX \\
&+\int_{D} a(X)u(X)U(X-\xi)dX \\
&=\int_{D} f(X)U(X-\xi)dX. \qquad (4)
\end{aligned}$$

Consider the following properties of Delta-Dirac function:

$$\int_{D} u(X)\delta(X-\xi)dX = \begin{cases} u(\xi); & \xi\in D, \\ \frac{1}{2}u(\xi); & \xi\in\Gamma=\partial D, \end{cases}$$

$$X=(x_1,x_2),\ \ \xi=(\xi_1,\xi_2),$$

and according to the definition of fundamental solution we have:

$$\frac{\partial U(X-\xi)}{\partial x_2}+i\frac{\partial U(X-\xi)}{\partial x_1}=\delta(X-\xi),$$

where $U(X-\xi)$ is fundamental solution of Cauchy-Riemann equation. By using these properties we have from

$$\begin{aligned}
&\int_{\Gamma} u(X)U(X-\xi)\big[\cos(\nu,x_2)+i\cos(\nu,x_1)\big]dX \\
&+\int_{D} a(X)u(X)U(X-\xi)dX-\int_{D} f(X)U(X-\xi)dX \\
=&\int_{D} u(X)\left[\frac{\partial U(X-\xi)}{\partial x_2}+i\frac{\partial U(X-\xi)}{\partial x_1}\right]dX \\
=&\int_{D} u(X)\delta(X-\xi)dX \\
=&\begin{cases} u(\xi); & \xi\in D, \\ \frac{1}{2}u(\xi); & \xi\in\Gamma. \end{cases} \qquad (5)
\end{aligned}$$

Note that the vector ν is outnormal vector to the boundary Γ. Now, regarding the second case of (5), we have the following relation

$$\begin{aligned}
\frac{1}{2}u(\xi) =& \int_{\Gamma} u(X)U(X-\xi)\big[\cos(\nu,x_2)+i\cos(\nu,x_1)\big]dX \\
&+\int_{D} a(X)u(X)U(X-\xi)dX-\int_{D} f(X)U(X-\xi)dX, \quad \xi\in\Gamma, \qquad (6)
\end{aligned}$$

and by considering the two parts of $\Gamma=\Gamma_1\cup\Gamma_2$ we have the following relations for Γ_1,Γ_2 respectively as shown in Figure 2 and Figure 3:

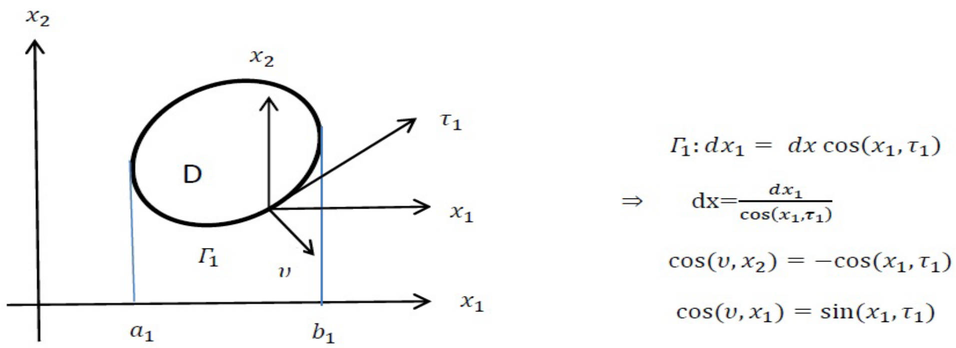

Figure 2.

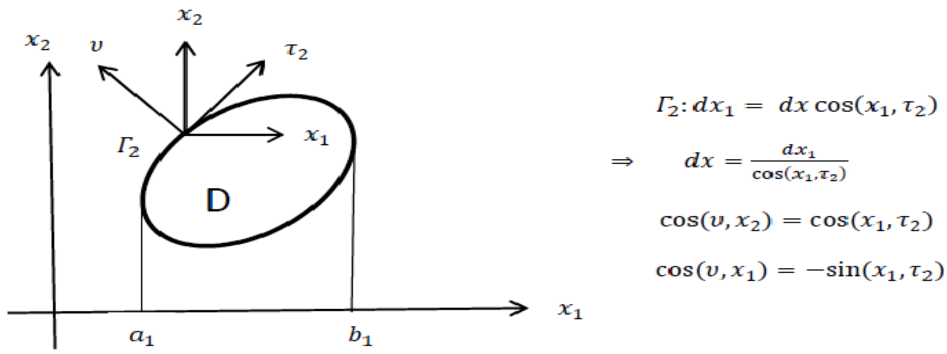

Figure 3.

$$
\begin{aligned}
&\frac{1}{2}u(\xi_1,\gamma_1(\xi_1)) = \\
&\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_1(\xi_1))\cdot\frac{-\cos(x_1,\tau_1)+i\sin(x_1,\tau_1)}{\cos(x_1,\tau_1)}dx_1 \\
&+\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))\cdot\frac{\cos(x_1,\tau_2)-i\sin(x_1,\tau_2)}{\cos(x_1,\tau_2)}dx_1 \\
&+\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX-\int_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX.
\end{aligned}
$$

Where τ_1, τ_2 are the tangent vectors to the boundary Γ_1, Γ_2, respectively. The above fractions in integrands can be written in the next form by using following relation:

$$\gamma_i'(x_1) = \tan(x_1,\tau_i) = \frac{\sin(x_1,\tau_i)}{\cos(x_1,\tau_i)}, \qquad i=1,2,$$

and

$$
\begin{aligned}
\frac{1}{2}u(\xi_1,\gamma_1(\xi_1)) &= -\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_1(\xi_1))\cdot(1-i\gamma_1'(x_1))dx_1 \\
&+\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))(1-i\gamma_2'(x_1))dx_1 \\
&+\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX \\
&-\int_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX. \qquad (7)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{1}{2}u(\xi_1,\gamma_2(\xi_1)) &= -\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1))\cdot(1-i\gamma_1'(x_1))dx_1 \\
&+\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_2(\xi_1))(1-i\gamma_2'(x_1))dx_1 \\
&+\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX \\
&-\int_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX. \qquad (8)
\end{aligned}
$$

Relations (7) and (8) are called compatibility conditions for equation (1).

3. Regularization

There are singularities in the expression of the fundamental solution. Hence we consider singularities only in the first term of (7) and the second term of (8). Other terms in these relations have no singularities. For more illumination these singularities, the first term of the relation (7) and the second term of the (8) are rewritten as follows:

$$
\begin{aligned}
(\text{first term of 7}) &= -\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_1(\xi_1))(1-i\gamma_1'(x_1))dx_1 \\
&= -\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))\cdot\frac{1}{2\pi}\frac{1-i\gamma_1'(x_1)}{\gamma_1(x_1)-\gamma_1(\xi_1)+i(x_1-\xi_1)}dx_1 \\
&= -\frac{1}{2\pi}\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))\cdot\frac{1-i\gamma_1'(x_1)}{\gamma_1'(\sigma_1(x_1,\xi_1))(x_1-\xi_1)+i(x_1-\xi_1)}dx_1 \\
&= -\frac{1}{2\pi}\int_{a_1}^{b_1} \frac{u(x_1,\gamma_1(x_1))}{x_1-\xi_1}\cdot\frac{1-i\gamma_1'(x_1)}{\gamma_1'(\sigma_1)+i}dx_1. \qquad (9)
\end{aligned}
$$

Hence, we consider singularities only in the first term of (7) and the second term of (8). Note that in the relation (9) the mean value theorem for derivative has been used [18], that is:

$$\gamma_1(x_1)-\gamma_1(\xi_1)=(x_1-\xi_1)\gamma_1'(\sigma_1(x_1,\xi_1)), \qquad x_1<\sigma_1<\xi_1.$$

Now in the kernel of the last term of (9), if the imaginary $i=\sqrt{-1}$ is added and subtracted, then we have:

$$-\frac{1}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{x_1-\xi_1}\left(\frac{1-i\gamma_1'(x_1)}{\gamma_1'(\sigma_1)+i}+i-i\right)dx_1 = \frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{x_1-\xi_1}dx_1$$
$$+\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{\gamma_1'(\sigma_1)+i}\cdot\frac{\gamma_1'(x_1)-\gamma_1'(\sigma_1)}{x_1-\xi_1}dx_1. \qquad (10)$$

Now, we suppose the curves γ_1,γ_2 satisfy in Hölder continuity, that is:

$$|\gamma_1'(x_1)-\gamma_1'(\sigma_1)|\le c|x_1-\sigma_1|^r, \qquad 0<r<1, \qquad c=constant. \qquad (11)$$

By applying inequality (11) in the relation (10), the singularity is converted to a weak singularity. Similarly, if the same process is applied for the second term of (8), we get:

$$\begin{aligned}\text{(second term of 8)} &= \int_{a_1}^{b_1}u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_2(\xi_1))\big(1-i\gamma_2'(x_1)\big)dx_1\\ &= -\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1\\ &\quad -\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(x_1)-\gamma_2'(\sigma_2)}{x_1-\xi_1}dx_1. \qquad (12)\end{aligned}$$

By considering Hölder continuity for the curve γ_2, the singularity in the denominator (12) is removed as same as operations mentioned above about the first term of (7). Now, the relations (10) and (12) are replaced in the (7) and (8) respectively, we have:

$$\frac{1}{2}u(\xi_1,\gamma_1(\xi_1))=\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{x_1-\xi_1}dx_1+\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{\gamma_1'(\sigma_1)+i}\cdot\frac{\gamma_1'(x_1)-\gamma_1'(\sigma_1)}{x_1-\xi_1}dx_1$$
$$+\int_{a_1}^{b_1}u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))(1-i\gamma_2'(x_1))dx_1$$
$$+\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX-\int_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX, \qquad (13)$$

and

$$\frac{1}{2}u(\xi_1,\gamma_2(\xi_1)) = -\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1 - \frac{i}{2\pi}\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(x_1)-\gamma_2'(\sigma_2)}{x_1-\xi_1}dx_1$$

$$-\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1))(1-i\gamma_1'(x_1))dx_1$$

$$+\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX - \int_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX. \qquad (14)$$

We see that there are singularities only in the terms of $\int_{a_1}^{b_1}\frac{u(x_1,\gamma_1(x_1))}{x_1-\xi_1}dx_1$ and $\int_{a_1}^{b_1}\frac{u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1$. For removing their singularities, we construct the following combination. For this, the relations (13) and (14) are multiplied by $\alpha_1(\xi_1)$ and $-\alpha_2(\xi_1)$ respectively, we have:

$$\frac{1}{2}\alpha_1(\xi_1)u(\xi_1,\gamma_1(\xi_1)) - \frac{1}{2}\alpha_2(\xi_1)u(\xi_1,\gamma_2(\xi_1))$$

$$= \frac{i}{2\pi}\int_{a_1}^{b_1}\frac{\alpha_1(\xi_1)u(x_1,\gamma_1(x_1))}{x_1-\xi_1}dx_1$$

$$+\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{\alpha_1(\xi_1)u(x_1,\gamma_1(x_1))}{\gamma_1'(\sigma_1)+i}\cdot\frac{\gamma_1'(x_1)-\gamma_1'(\sigma_1)}{x_1-\xi_1}dx_1,$$

$$+\alpha_1(\xi_1)\int_{a_1}^{b_1} u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))(1-i\gamma_2'(x_1))dx_1$$

$$+\alpha_1(\xi_1)\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX$$

$$-\alpha_1(\xi_1)\int_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX$$

$$+\alpha_2(\xi_1)\int_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1))(1-i\gamma_1'(x_1))dx_1$$

$$+\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{\alpha_2(\xi_1)u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1$$

$$+\frac{i}{2\pi}\int_{a_1}^{b_1}\frac{\alpha_2(\xi_1)u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(x_1)-\gamma_2'(\sigma_2)}{x_1-\xi_1}dx_1$$

$$-\alpha_2(\xi_1)\int_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX$$

$$+\alpha_2(\xi_1)\int_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX. \qquad (15)$$

Now for removing the singularities, in the first and seventh terms of (15), $\int\limits_{a_1}^{b_1} \frac{\alpha_1(\xi_1)u(x_1,\gamma_1(x_1))}{x_1-\xi_1}dx_1$ and $\int\limits_{a_1}^{b_1} \frac{\alpha_2(\xi_1)u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1$ the functions $\alpha_1(x_1)$ and $\alpha_2(x_1)$ are added and subtracted in their numerators respectively. Therefor the relation (15) reduces to the new form:

$$\alpha_1(\xi_1)u(\xi_1,\gamma_1(\xi_1)) - \alpha_2(\xi_1)u(\xi_1,\gamma_2(\xi_1))$$

$$= \frac{i}{\pi}\int\limits_{a_1}^{b_1} \frac{\alpha_1(\xi_1)-\alpha_1(x_1)}{x_1-\xi_1}u(x_1,\gamma_1(x_1))dx_1$$

$$+\frac{i}{\pi}\int\limits_{a_1}^{b_1} \frac{\alpha_2(\xi_1)-\alpha_2(x_1)}{x_1-\xi_1}u(x_1,\gamma_2(x_1))dx_1$$

$$-\frac{i}{\pi}\int\limits_{a_1}^{b_1} ln|x_1-\xi_1|\alpha'(x_1)dx_1$$

$$+\frac{i}{\pi}\int\limits_{a_1}^{b_1} \frac{\alpha_1(\xi_1)u(x_1,\gamma_1(x_1))}{\gamma_1'(\sigma_1)+i}\cdot\frac{\gamma_1'(x_1)-\gamma_1'(\sigma_1)}{x_1-\xi_1}dx_1$$

$$+2\alpha_1(\xi_1)\int\limits_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX$$

$$-2\alpha_1(\xi_1)\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX+$$

$$+2\alpha_1(\xi_1)\int\limits_{a_1}^{b_1} u(x_1,\gamma_2(x_1))U(x_1-\xi_1,\gamma_2(x_1)-\gamma_1(\xi_1))(1-i\gamma_2'(x_1))dx_1$$

$$+2\alpha_2(\xi_1)\int\limits_{a_1}^{b_1} u(x_1,\gamma_1(x_1))U(x_1-\xi_1,\gamma_1(x_1)-\gamma_2(\xi_1))(1-i\gamma_1'(x_1))dx_1$$

$$+\frac{i}{\pi}\int\limits_{a_1}^{b_1} \frac{\alpha_2(\xi_1)u(x_1,\gamma_2(x_1))}{\gamma_2'(\sigma_2)+i}\cdot\frac{\gamma_2'(x_1)-\gamma_2'(\sigma_2)}{x_1-\xi_1}dx_1$$

$$-2\alpha_2(\xi_1)\int\limits_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX$$

$$+2\alpha_2(\xi_1)\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX.$$

Remark 3.1. *Note that the logarithmic term of the above relation (16) is given following calculations:*

$$\int\limits_{a_1}^{b_1} \frac{\alpha_1(x_1)u(x_1,\gamma_1(x_1))+\alpha_2(x_1)u(x_1,\gamma_2(x_1))}{x_1-\xi_1}dx_1$$

$$= \int_{a_1}^{b_1} \frac{\alpha(x_1)}{x_1 - \xi_1} dx_1 = \int_{a_1}^{b_1} \alpha(x_1) d\left(\ln|x_1 - \xi_1|\right)$$

$$= \alpha(x_1) \ln|x_1 - \xi_1| \Big|_{x_1 = a_1}^{b_1} - \int_{a_1}^{b_1} \ln|x_1 - \xi_1| \alpha'(x_1) dx_1$$

$$= \alpha(b_1) \ln|b_1 - \xi_1| - \alpha(a_1) \ln|a_1 - \xi_1| - \int_{a_1}^{b_1} \ln|x_1 - \xi_1| \alpha'(x_1) dx_1.$$

Supposing $\alpha(b_1) = \alpha(a_1) = 0$, and $\alpha(x_1) \in C^{(1)}(a_1, b_1)$, we imply the following result:

$$\int_{a_1}^{b_1} \frac{\alpha_1(x_1)u(x_1, \gamma_1(x_1)) + \alpha_2(x_1)u(x_1, \gamma_2(x_1))}{x_1 - \xi_1} dx_1 =$$

$$\int_{a_1}^{b_1} \frac{\alpha(x_1)}{x_1 - \xi_1} dx_1 = -\int_{a_1}^{b_1} \ln|x_1 - \xi_1| \alpha'(x_1) dx_1. \tag{16}$$

Note that to obtain the above results, the boundary condition (2) has been used. Now, we consider the relation (16) with boundary condition (2) together as an algebraic system as follows:

$$\begin{cases} \alpha_1(\xi_1)u(\xi_1, \gamma_1(\xi_1)) + \alpha_2(\xi_1)u(\xi_1, \gamma_2(\xi_1)) = \alpha(\xi_1), \\ \alpha_1(\xi_1)u(\xi_1, \gamma_1(\xi_1)) - \alpha_2(\xi_1)u(\xi_1, \gamma_2(\xi_1)) = -\dfrac{i}{\pi} \displaystyle\int_{a_1}^{b_1} ln|x_1 - \xi_1| \alpha'(x_1) dx_1 + A(\xi_1), \end{cases} \tag{17}$$

where

$$A(\xi_1) = \frac{i}{\pi} \int_{a_1}^{b_1} \frac{\alpha_1(\xi_1) - \alpha_1(x_1)}{x_1 - \xi_1} u(x_1, \gamma_1(x_1)) dx_1 + \frac{i}{\pi} \int_{a_1}^{b_1} \frac{\alpha_2(\xi_1) - \alpha_2(x_1)}{x_1 - \xi_1} u(x_1, \gamma_2(x_1)) dx_1$$

$$+ \frac{i}{\pi} \int_{a_1}^{b_1} \frac{\alpha_1(\xi_1)u(x_1, \gamma_1(x_1))}{\gamma_1'(\sigma_1) + i} \cdot \frac{\gamma_1'(x_1) - \gamma_1'(\sigma_1)}{x_1 - \xi_1} dx_1$$

$$+ 2\alpha_1(\xi_1) \int_D a(X)u(X)U(x_1 - \xi_1, x_2 - \gamma_2(\xi_1)) dX$$

$$- 2\alpha_1(\xi_1) \int_D f(X)U(x_1 - \xi_1, x_2 - \gamma_1(\xi_1)) dX +$$

$$+ 2\alpha_1(\xi_1) \int_{a_1}^{b_1} u(x_1, \gamma_2(x_1))U(x_1 - \xi_1, \gamma_2(x_1) - \gamma_1(\xi_1))(1 - i\gamma_2'(x_1)) dx_1$$

$$+ 2\alpha_2(\xi_1) \int_{a_1}^{b_1} u(x_1, \gamma_1(x_1))U(x_1 - \xi_1, \gamma_1(x_1) - \gamma_2(\xi_1))(1 - i\gamma_1'(x_1)) dx_1$$

$$+ \frac{i}{\pi} \int_{a_1}^{b_1} \frac{\alpha_2(\xi_1)u(x_1, \gamma_2(x_1))}{\gamma_2'(\sigma_2) + i} \cdot \frac{\gamma_2'(x_1) - \gamma_2'(\sigma_2)}{x_1 - \xi_1} dx_1$$

$$-2\alpha_2(\xi_1)\int\limits_D a(X)u(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX$$

$$p+2\alpha_2(\xi_1)\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX.$$

By solving this algebraic system, the following relations for boundary values of unknown function $u(\xi_1,\gamma_1(\xi_1))$ and $u(\xi_1,\gamma_2(\xi_1))$ are obtained:

$$\begin{cases} u(\xi_1,\gamma_1(\xi_1))=\dfrac{\alpha(\xi_1)}{2\alpha_1(\xi_1)}+\dfrac{1}{2\alpha_1(\xi_1)}\big(-\dfrac{i}{\pi}\int\limits_{a_1}^{b_1} ln|x_1-\xi_1|\alpha'(x_1)dx_1+A(\xi_1)\big),\\ u(\xi_1,\gamma_2(\xi_1))=\dfrac{\alpha(\xi_1)}{2\alpha_2(\xi_1)}-\dfrac{1}{2\alpha_2(\xi_1)}\big(-\dfrac{i}{\pi}\int\limits_{a_1}^{b_1} ln|x_1-\xi_1|\alpha'(x_1)dx_1+A(\xi_1)\big).\end{cases}$$

The relations (17) are as a aystem of second Fredholm integral equations with regularized (with weak singularity) kernels for boundary values of main unknown function. For simplicity, we can rewrite the relations (17) in the compact form:

$$\begin{cases} u(\xi_1,\gamma_1(\xi_1))=\int\limits_{a_1}^{b_1}[H_{11}(\xi_1,\eta_1)u(\eta_1,\gamma_1(\eta_1))+H_{12}(\xi_1,\eta_1)u(\eta_1,\gamma_2(\eta_1))]d\eta_1+f_1(\xi_1),\\ u(\xi_1,\gamma_2(\xi_1))=\int\limits_{a_1}^{b_1}[H_{21}(\xi_1,\eta_1)u(\eta_1,\gamma_1(\eta_1))+H_{22}(\xi_1,\eta_1)u(\eta_1,\gamma_2(\eta_1))]d\eta_1+f_2(\xi_1),\end{cases} \tag{18}$$

where H_{11},H_{12} ,H_{21} and H_{22} are the kernels of integral terms over the interval $[a_1,b_1]$ and $f_1(\xi_1)$, $f_2(\xi_2)$ are given as follws:

$$f_1(\xi_1)=\frac{\alpha(\xi_1)}{2\alpha_1(\xi_1)}-\frac{i}{2\pi}\int\limits_{a_1}^{b_1}\frac{ln|x_1-\xi_1|\alpha'(x_1)}{\alpha_1(\xi_1)}dx_1$$

$$-\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX$$

$$-\frac{\alpha_2(\xi_1)}{\alpha_1(\xi_1)}\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX,$$

and

$$f_2(\xi_1)=\frac{\alpha(\xi_1)}{2\alpha_2(\xi_1)}+\frac{i}{2\pi}\int\limits_{a_1}^{b_1}\frac{ln|x_1-\xi_1|\alpha'(x_1)}{\alpha_2(\xi_1)}dx_1$$

$$+\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_2(\xi_1))dX$$

$$-\frac{\alpha_1(\xi_1)}{\alpha_2(\xi_1)}\int\limits_D f(X)U(x_1-\xi_1,x_2-\gamma_1(\xi_1))dX.$$

Remark 3.2. *If we use the following changes is variables:*

$$u(\xi_1)=\begin{pmatrix} u(\xi_1,\gamma_1(\xi_1)\\ u(\xi_1,\gamma_2(\xi_1)\end{pmatrix}, H(\xi_1,\eta_1)=\begin{pmatrix} H_{11} & H_{12}\\ H_{21} & H_{22}\end{pmatrix}, F(\xi_1)=\begin{pmatrix} f_1(\xi_1)\\ f_2(\xi_1)\end{pmatrix},$$

then we can rewrite relation (18) in matrix form:

$$u(\xi_1) = F(\xi_1) + \int_{a_1}^{b_1} H(\xi_1, \eta_1) u(\eta_1) d\eta_1.$$

By considering calculations and relations in Section 2, we result the following Theorem 1.1. and by considering the regularization in section 3, Remark 3.1 and solving the algebraic system (17) with Remark 3.2 we result the following Theorem 2.

4. Main results

Theorem 4.1. *In the boundary value problem (1)-(2), suppose $D \subset \mathbb{R}^2$ is a bounded domain with Lyapunov boundary line, that is:*

$$\partial D = \Gamma = \Gamma_1 \cup \Gamma_2, \ x_2 = \gamma_k(x_1) \ k = 1, 2$$

also $\alpha_i(x_1); i = 1, 2$ and $\alpha(x_1)$ are Hölder functions and $\alpha(a_1) = \alpha(b_1) = 0$. Then every solution of boundary value problem (1)-(2) satisfies in regularized relations (18).

Theorem 4.2. *Under the hypothesis of Theorem* **??***, the boundary values of an unknown function $u(\xi_1, \xi_2)$ of BVP (1)-(2) satisfy in the second kind Fredholm boundary integral equation:*

$$u(\xi_1) = F(\xi_1) + \int_{a_1}^{b_1} H(\xi_1, \eta_1) u(\eta_1) d\eta_1,$$

and its solution can be written as

$$u(\xi_1) = F(\xi_1) + \int_{a_1}^{b_1} R(\xi_1, \eta_1) F(\eta_1) d\eta_1,$$

where $R(\xi_1, \eta_1)$ is a resolvent kernel of Fredholm integral equation and $H(\xi_1, \eta_1)$ is a regularized kernel [19].

5. Conclusion

There are many different methods and several approaches to solving boundary value problems. However, solving these problems are usually tricky. This article presents a new approach for solving BVPs. According to this method, the given BVP to the second kind regularized Fredholm boundary integral equations were reduced. In other words, for boundary values of unknown function, a boundary integral equation which their kernels had no singularities were obtained. By using these boundary values of unknown function and substituting them in the first case of relation(5), an analytic solution for the given BVP (1)-(2)was obtained.

References

[1] Aksoy, U., Okayc, A., (2010), A survey on boundary value problems for complex partial differential equations, Adv. Dyn. Syst. Appl., 5(2), pp.133-158.

[2] Aliev, N., Jahanshahi, M., (1997), Sufficent conditions for reduction of the BVP including a mixed P.D.E with nonlocal boundary conditions to Fredholm integral equation, JNT. J. Math. Educ. Tech. Scie, 28(3), pp.419-425.

[3] Aliev, N., Jahanshahi, M., Hosseini, S.M., (2005), An analytical-numerical method for investigation and solving three dimensional steady state Navier-Stokes equations, II Int. J. Differ. Equ. Appl., 10(2), pp.135-145.

[4] Aliev, N., Rezapour, Sh., Jahanshahi, M., (2009), On a mixed problem for Navier-Stokes System in the unit cube, Math. Morav., 13(1), pp.13-24.

[5] Begehr, H., (2005), Boundary value problems in complex analysis, I. Bol. Asoc. Mat. Venez., XII(1), pp.65-85.

[6] Cheraghi Tofigh, A.A., Ezzati, R., (2019), Introducing a new approach to solve nonlinear Volterra-Fredholm integral equations, TWMS J. Pure Appl. Math., 10(2), pp.175-187

[7] Courant, R., Hilbert, D., (1962), Methods of Mathematical Physics, Interscience Publishers, 576p.

[8] Evans, L.C., (1989), Partial Differential Equations, American Mathematical Society, Providence, Rhode Island, 662p.

[9] Jahanshahi, M., Aliev, N., (2004), Determining of an analytic function on its analytic domain by Cauchy-Riemann equation with special kind of boundary conditions, Southeast Asian Bull. Math., 28(1), pp.33-39.

[10] Jahanshahi, M., Aliev, N., Hosseini, S.M., (2009), An analytic method for investigation and solving two-dimensional steady state Navier-Stokes equations (I), Southeast Asian Bull. Math., 33(6), pp.1075-1089.

[11] Jahanshahi, M., Fatehi, M., (2008), Analytic solution for the Cauchy-Riemann equations in the first quarter, Int. J. Pure Appl. Math., 46(2), pp.245-259.

[12] Jahanshahi, M., Sajjadmanesh, M., (2013), Analytical solutions for the Stepjen's inverse problem with local boundary conditions including a first order hyperbolic equation, Bull. Iranian Math. Soc., 39(5), pp.855-864.

[13] Kondo, J., (1991), Integral Equations, Oxford University Press.

[14] Lesnic, D., (2009), On the boundary integral equations for a two-dimensional slowly rotating highly viscous flow, Adv. Appl. Math. Mech., 1(1), pp.140-150.

[15] Mahmoodi Darani, N., Maleknejad, Kh., Mesgarani, H. (2019), A new approach for two-dimensional nonlinear mixed Volterra-Fredholm integral equations and its convergence analysis, TWMS J. Pure Appl. Math., 10(1), pp.132-139

[16] Myint-U, T., Debnat, L., (2007), Linear Partial Differential Equations for Scientists and Engineers, Fourth Edition, Birkhauser Boston Publisher, 778p.

[17] Rasulov, A.B., Soldatov, A.P., (2016), Boundary value problem for a generalized Cauchy-Riemann equation with singular coefficients, Diff. Equ., 5(52), pp.616-629.

[18] Rudin, W., (1974), Principal of Mathematical Analysis, Third Edition by McGraw-Hill, 342p.

[19] Sajjadmanesh, M., Jahanshahi, M., Aliev, N., (2013), Tikhonov-Lavrentev type inverse problem including Cauchy-Riemann equation, Azerb. J. Math., 3(1), pp.104-110.

[20] Vladimirov, V.S., (1984), Equations of Mathematical Physics, Mir Publishers, Moscow,464p.

[21] Wang, Y., Wang, Y., (2012), Two boundary-value problems for the Cauchy-Riemann equation in a sector, Complex Anal. Oper. Theory, 6(6), pp.1121-1138.

Javad Ebadpour Golanbar - is a Ph.D. student in the field of applied mathematics (differential equations) at Azerbaijan Shahid Madani University since 2016. His research interests are elliptic partial differential equations and second order Fredholm integral equations.

Nihan Aliev - was born in 1938 in Baku, Azerbaijan. He worked as a Professor in Islamic Republic of Iran in 1993-2006 years. His research interests are elliptic partial differential equations and second order Fredholm integral equations.

Mohammad Jahanshahi - has got Ph.D. degree from Tarbiat Modares University, Tehran in 2000 and now he is a full Professor at the Department of Mathematics in Azerbaijan Shahid Madani University, Tabriz, Iran. His research interests include study of existence and uniqueness of solution of initial and boundary value problems including ordinary and partial differential equations.